A Novel Method for Credit Portfolio Analysis - A Numerical Approximation Approach -

Mizuho-DL Financial Technology
Outline

I. What is “Analytical Method”?  
II. Laplace inversion method  
III. High-speed calculation of MGF
1. What is “Analytical Method”?
I. What is “Analytical Method”?

- Challenges in credit risk measurement technology
  - Fulfilling high-speed and high-accuracy at the same time
    - It is easy to satisfy either one and sacrifice the other.
  - Excluding simulation error
    - Monte Carlo (MC) simulation, which is the most familiar measurement method, inevitably involves statistical error.
  - Calculating risk contributions (RC) accurately
    - High-accuracy RC calculation method has not yet been established.
  - Handling multi-factor models
    - It is difficult to deal with multi-factor models by methods other than MC simulation.
I. What is “Analytical Method”?

Overview of credit risk measurement approaches

- MC simulation
  - Advantages: Allows for high-accuracy calculation (at the cost of spending much time)
    Applicable to multi-factor models
  - Disadvantages: Time-consuming to gain high-accuracy
    Difficult to calculate RC

- Approximate solution (IRB in Basel II, saddle-point…)
  - Advantages: High-speed in general
    Can be applied to RC calculation
  - Disadvantages: Accuracy depends on portfolio
    Difficult to handle multi-factor models
I. What is “Analytical Method”?

Choice of approach

- There is little room for improving the two approaches.
  - In MC simulation, it is difficult to quickly calculate RC.
    - Of course, it is easy to speed up MC simulation if you can afford expensive servers...
  - In approximate solution, accuracy worsens for a certain class of portfolios (e.g. concentrated portfolios).
    - Are there any approximation formula that is highly accurate for arbitrary loss distributions?

We take the 3rd approach: numerical approximation
I. What is “Analytical Method”?

What is numerical approximation?

- Only numerically approximate the exact expression of loss distribution function and RC
  - “Analytical Method” developed by Mizuho-DL FT is a specific example of the numerical approximation approach
- Free from systematic approximation based on unrealistic assumptions on portfolio (e.g. complete diversification)
- A sophisticated calculation process enables high-speed measurement.
  - Calculation time can be substantially reduced by taking “discretization” approximation in calculation process.

An approach essentially different from MC simulation and approximate solution
1. What is “Analytical Method”? 

Performance study of “Analytical Method”

Sample portfolios used for the study

<table>
<thead>
<tr>
<th>Number of obligors</th>
<th>1000, 10000, 1000000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distribution of exposures</td>
<td>Homogeneous</td>
</tr>
<tr>
<td></td>
<td>Power law (Concentrated)</td>
</tr>
<tr>
<td>Number of sectors</td>
<td>33</td>
</tr>
<tr>
<td>Number of ratings</td>
<td>19</td>
</tr>
</tbody>
</table>

PC spec

CPU: Core™2 Duo 2.66GHz, Memory: 3.12GB

Ordinary desktop PC

(For more details, see our paper “A novel methodology for credit portfolio analysis ...”)

Mizuho-DL Financial Technology
I. What is “Analytical Method”?

Calculation time in “Analytical Method”

The following table summarizes risk measurement time in “Analytical Method”.

Table 1: Calculation time in “Analytical Method”

<table>
<thead>
<tr>
<th>Number of obligors</th>
<th>Distribution function (including VaR and CVaR)</th>
<th>RC of all obligors</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,000</td>
<td>40sec</td>
<td>2min43sec</td>
</tr>
<tr>
<td>10,000</td>
<td>42sec</td>
<td>2min51sec</td>
</tr>
<tr>
<td>1,000,000</td>
<td>3min45sec</td>
<td>17min43sec</td>
</tr>
</tbody>
</table>

(Distribution of exposures : Concentrated)
I. What is “Analytical Method”? 

Calculation accuracy in “Analytical Method” 

The following table summarizes risk measurement accuracy in “Analytical Method”.

Table 2: Calculation accuracy in “Analytical Method”

<table>
<thead>
<tr>
<th>Risk measure</th>
<th>Confidence level</th>
<th>Error relative to MC with $10^7$ scenarios</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Homogeneous</td>
</tr>
<tr>
<td>VaR</td>
<td>99.00%</td>
<td>0.32%</td>
</tr>
<tr>
<td></td>
<td>99.90%</td>
<td>0.31%</td>
</tr>
<tr>
<td></td>
<td>99.97%</td>
<td>-0.22%</td>
</tr>
<tr>
<td>CVaR</td>
<td>99.00%</td>
<td>0.30%</td>
</tr>
<tr>
<td></td>
<td>99.90%</td>
<td>-0.21%</td>
</tr>
<tr>
<td></td>
<td>99.97%</td>
<td>-0.78%</td>
</tr>
</tbody>
</table>

(Number of obligors=1000000)
I. What is “Analytical Method”?

Performance curve of the three approaches

- MC simulation: Extremely steep curve
- Approx. solution: Always fast but approximation is crude
- Numerical approx. ( "Analytical Method"): Gentle curve

Calculation time

- Slow: MC simulation
  - Dependent on number of scenarios
- Fast: Approx. solution
  - Dependent on adjustable parameters
- True distribution: Numerical approx.
  - Accuracy

Dependent on number of scenarios
1. What is “Analytical Method”? 

Calculation accuracy in far tail region

The figures below magnify the region near the 99.9% VaR of a loss distribution.
I. What is “Analytical Method”? 

Allocating risk measures into individual obligors

- Individual VaR and CVaR contributions are defined as follows.

\[ R_{i}^{VaR} = \frac{\partial VaR}{\partial E_i} \quad , \quad R_{i}^{CVaR} = \frac{\partial CVaR}{\partial E_i} \]

- VaR contribution: \( EC_{i}^{VaR} = E_i \cdot R_{i}^{VaR} \)
- CVaR contribution: \( EC_{i}^{CVaR} = E_i \cdot R_{i}^{CVaR} \)

These contributions can be accurately calculated by “Analytical Method”, which provides much information about far tail region.
1. What is “Analytical Method”?

Allocating risk measures into individual obligors

The graphs below show the cumulative exposures and VaR contributions (at 99.9% confidence level) of a concentrated portfolio with 10 thousand obligors.

Concentration risk can be quantitatively studied.
I. What is “Analytical Method”?

Use of “Analytical Method” in banking practice

“Analytical Method” is quite effective in “comparing risk measures”, because it is free from simulation error.

For example, impact study can be done in the following analyses without being disturbed by simulation error.

- Stress testing for various scenarios
- Analysis of time series variation of risk measures
- Buying and selling simulation of loans
I. What is “Analytical Method”?

Buying and selling simulation

- Analyze the impact of buying a new loan on risk measures of a portfolio with 10 thousand obligors

  Loan amount : ¥10bn (0.1% of total exposure)
  PD : 1.0%
I. What is “Analytical Method”?

Buying and selling simulation

The following tables show the impact of new loan.

MC with 1 million scenarios

<table>
<thead>
<tr>
<th>Risk measure</th>
<th>Before buying</th>
<th>After buying</th>
<th>Difference</th>
<th>Rate of change</th>
</tr>
</thead>
<tbody>
<tr>
<td>EL</td>
<td>170,263,758,805</td>
<td>170,433,816,363</td>
<td>170,557,558</td>
<td>0.100%</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>217,798,595,633</td>
<td>217,752,755,326</td>
<td>-45,840,307</td>
<td>-0.021%</td>
</tr>
<tr>
<td>VaR 99.00%</td>
<td>931,651,036,249</td>
<td>931,090,171,646</td>
<td>-560,864,603</td>
<td>-0.060%</td>
</tr>
<tr>
<td>VaR 99.97%</td>
<td>1,399,855,268,456</td>
<td>1,399,098,926,498</td>
<td>-756,342,258</td>
<td>-0.054%</td>
</tr>
<tr>
<td>CVaR 99.00%</td>
<td>1,138,453,814,342</td>
<td>1,135,207,577,284</td>
<td>-3,246,237,058</td>
<td>-0.285%</td>
</tr>
<tr>
<td>CVaR 99.97%</td>
<td>1,995,759,462,233</td>
<td>1,959,043,201,218</td>
<td>-36,716,261,015</td>
<td>-1.840%</td>
</tr>
</tbody>
</table>

"Analytical Method" can detect a tiny difference.
I. What is “Analytical Method”?

- Formula for portfolio loss distribution
  - Probability density function (PDF) can be calculated as the Laplace inversion of moment generating function (MGF).

\[ f_L(t) = \frac{1}{2\pi} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\alpha t} \left\{ \int_{-\infty}^{\infty} \prod_{i=1}^{N} \left( 1 - p_i(\bar{x}) + p_i(\bar{x})e^{-\alpha E_i} \right) \phi(\bar{x}) d\bar{x} \right\} d\alpha \]
1. What is “Analytical Method”? 

Some crucial points in PDF calculation

<Point 1> MGF calculation method

- Need to quickly and accurately calculate (multi-dimensional) integral w.r.t. common risk factors

<Point 2> Laplace inversion method

- Need to quickly and accurately calculate integral along an infinite interval on the complex plane

“Analytical Method” is a numerical approximation for quickly and accurately executing calculations in <Point 1> and <Point 2>. 
II. Laplace inversion method
II. Laplace inversion method

Idea of Laplace inversion

In “Analytical Method”, Laplace inversion is done by the following two steps.

⟨L-1⟩
Represent the integral in Laplace inversion formula by an infinite series

⟨L-2⟩
Calculate the infinite series using a convergence acceleration method
II. Laplace inversion method

<L-1> The Poisson method

In the Poisson method, the infinite integral in Laplace inversion formula is approximated by an infinite series.

\[
f_L(t) \approx \frac{h}{2\pi} \left[ \sum_{k=-\infty}^{\infty} \exp[(\gamma + ikh)t] \hat{f}_L(\gamma + ikh) \right]
\]

\[
= \frac{h}{\pi} \exp(\gamma t) \cdot \left[ \frac{\hat{f}_L(\gamma)}{2} + \sum_{k=1}^{\infty} \text{Re}\left\{ \hat{f}_L(\gamma + ikh) \exp[ikht] \right\} \right]
\]

\( h \): discretization width of integration interval

The Poisson method is formally identical with the trapezoidal formula.
11. Laplace inversion method

-  <L-1> The Poisson method
  - The Poisson method often gives a high-accuracy approximation.
  
  Error of the method when applied to distribution function
  \[ |err| \leq 1 - F\left(\frac{2\pi}{h} - t\right), \quad (0 < t \leq \frac{\pi}{h}) \]

  This formula is valid only when an infinite number of terms are completely summed up.

  In practice, the infinite series must be truncated, while the convergence of the series is sometimes very slow.

  A convergence acceleration method is necessary to improve the convergence.
11. Laplace inversion method

Illustration of “convergence acceleration”

<Example: $S_n = \sum_{k=1}^{n} \frac{(-1)^k}{k}$>

Sn is a very slowly converging series, but...

A few initial terms are enough to forecast the limit.

Possible to improve convergence?
11. Laplace inversion method

<L-2> About continued fraction (CF)

In “Analytical Method”, the infinite series is quickly calculated using the property of continued fraction.

Example: Continued fraction representation of arctan(x)

Infinite series rep.: \( \arctan x = x \left\{ 1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k}}{2k + 1} \right\} \)

Continued fraction rep.: \( \arctan x = \frac{x^2}{1 + \frac{x^2}{3 + \frac{(2x)^2}{7 + \cdots}}} \)
II. Laplace inversion method

<L-2> The property of continued fraction

Continued fraction representation sometimes converges extremely faster than infinite series representation does.

<Example: Numerical calculation of π>
Calculate an approximate value of π from π = 4arctan(1)

\[ \pi = 3.141592653589... \] (infinite series with 1 million terms)

\[ \pi = 3.1415926535898... \] (18th order continued fraction)
11. Laplace inversion method

<L-2> Convergence acceleration with CF

Convergence is expected to drastically improve if the infinite series can be converted to a continued fraction.

\[ f_L(t) \approx \frac{h}{\pi} \exp(\gamma t) \sum_{k=0}^{\infty} \text{Re}(c_k z^k) \]

\[ = \frac{h}{\pi} \exp(\gamma t) \times \left[ a_0 + \frac{a_1 z}{1 + \frac{a_2 z}{1 + \frac{a_3 z}{1 + \frac{a_4 z}{1 + \cdots}}} \right] \]

Need to calculate \( \{a_0, a_1, \cdots\} \), given the values of \( \{c_0, c_1, \cdots\} \)
II. Laplace inversion method

**<L-2> QD algorithm**

QD algorithm is a method for efficiently calculating the coefficients \( \{a_0, a_1, \cdots\} \) of a continued fraction from the coefficients \( \{c_0, c_1, \cdots\} \) of the corresponding infinite series.

**Input:**

\[ e_0^{(i)} = 0 \quad , \quad q_1^{(i)} = \frac{c_{i+1}}{c_i} \]

**Recursion:**

\[ e_k^{(i)} = e_{k-1}^{(i+1)} + q_k^{(i+1)} - q_k^{(i)} \quad , \quad q_k^{(i)} = \frac{q_k^{(i+1)} \cdot e_k^{(i+1)}}{e_k^{(i)}} \]

**Output:**

\[ a_{2k} = -e_k^{(0)} \quad , \quad a_{2k-1} = -q_k^{(0)} \]
### 11. Laplace inversion method

**<L-2> Convergence acceleration with CF**

Loss PDF can be quickly calculated by applying the property of continued fraction to Laplace inversion.

**<The result of the Poisson method>**

\[
\hat{f}(t) \approx \frac{h}{\pi} \exp(\gamma t) \cdot \left[ \frac{\hat{f}(\gamma)}{2} + \sum_{k=1}^{\infty} \text{Re} \{ \hat{f}(\gamma + ikh) \exp[ikht] \} \right] 
\]

Can be converted to a continued fraction by using QD algorithm with \( c_k = \hat{f}_L(\gamma + ikh) \), \( z = \exp(iht) \)
11. Laplace inversion method

<L-2> Calculation of loss distribution

Once the coefficients \( \{a_0, a_1, \cdots \} \) are obtained, the values of density function at arbitrary losses can be calculated.

\[
f(t) \approx \frac{h}{\pi} \exp(\gamma t) \cdot \left[ a_0 + \frac{a_1 z}{1 + \frac{a_2 z}{1 + \cdots}} \right]
\]

\[
z = \exp(iht)
\]

Since \( \{a_0, a_1, \cdots \} \) does not depend on loss \( t \), the whole distribution can be quickly calculated.
11. Laplace inversion method

Summary of Laplace inversion method

In “Analytical Method”, Laplace inversion is quickly and accurately done by the following two steps.

<L-1> Create an infinite series by the Poisson method

- The Poisson method is formally identical with the trapezoidal formula.

<L-2> Convergence acceleration by QD method

- QD method is an algorithm for continued fraction transformation.
III. High-speed calculation of MGF
III. High-speed calculation of MGF

- Conditions necessary for high-speed calculation
  - Conditionally independent
    - Conditional on fixed risk factors, default occurs independently.

- 1-factor decomposable
  - Portfolio consists of a small number of sub-portfolios, each of which is described by a 1-factor model.

These properties are the keys to high-speed calculation.
III. High-speed calculation of MGF

Conditional independence

“Analytical Method” can be applied to credit risk measurement based on conditional independence model.

For example, the multi-factor firm-value model below satisfies conditional independence.

\[ Z_i = \sum_{j=1}^{N_F} \alpha_{ij} X_j + \sqrt{1 - \sum_{j=1}^{N_F} \alpha_{ij}^2 \varepsilon_i} \]

- Firm-value
- Common risk factor
- Idiosyncratic factor
- Sensitivity coefficient
III. High-speed calculation of MGF

1-factor decomposition

In a standard practice, sensitivity coefficients are estimated only for “sectors”, not for individual obligors.

\[ \alpha_{ij} \quad \Rightarrow \quad \alpha_{S(i)j} \quad S(i) : \text{sector of obligor } i \]

In this case, the firm-value takes the form

\[ Z_i = \beta_{S(i)} \tilde{Y}^{S(i)} + \sqrt{1 - \beta_{S(i)}^2} \tilde{\varepsilon}_i \]

\[ \beta_{S(i)} = \sqrt{\sum_{j=1}^{N} \alpha_{S(i)j}^2} , \quad \tilde{Y}^{S(i)} = \frac{1}{\beta_{S(i)}} \sum_{j=1}^{N} \alpha_{S(i)j} \tilde{X}_j : \text{sectoral factor} \]

1-factor decomposable
III. High-speed calculation of MGF

- Idea of high-speed calculation of MGF
  - It is time-consuming to calculate MGF because it requires a loop proportional to the number of obligors.
  - This calculation is done for each sector on 1 dimension.

- Correlation structure is described among sectors.
  - High-dimensional calculation is necessary to take into account correlation.

Split marginal distribution calculation and correlation structure calculation
III. High-speed calculation of MGF

Illustration of high-speed calculation of MGF

1. Sectoral MGF is calculated in advance only at discrete points

2. Determine sectoral factors from integration points

3. Choose the nearest sectoral MGF

4. Sum up distribution

Sectoral factor of sector 1

Sectoral factor of sector 2
III. High-speed calculation of MGF

Illustration of high-speed calculation of MGF

5. Repeat 1.~4. for all integration points
III. High-speed calculation of MGF

Computational complexity of high-speed calculation

Computational complexity of a naive method for calculating MGF is given by

\[(\text{Number of obligors}) \times (\text{Number of integration points})\].

\[\sim O(10^{4-6}) \quad \sim O(10^6)\]

Computational complexity of the high-speed calculation is given by

\[\sim O(10^{4-6}) \quad \sim O(10^2)\]

\[(\text{Number of obligors}) \times (\text{Number of discrete points}) + (\text{Number of integration points}) \times (\text{Number of sectors})\].

\[\sim O(10^6) \quad \sim O(10^{0-2})\]

Computational complexity can be drastically reduced by the high-speed calculation.
III. High-speed calculation of MGF

Accuracy of high-speed calculation

In the high-speed calculation, the continuous Euclidean space spanned by sectoral factors is replaced in effect with a hypercubic discrete lattice.

The error due to this approximation can be controlled, and becomes small as the number of discrete points (which specifies the fineness of the lattice) increases.

We have confirmed that this approximation was sufficiently accurate if the number of discrete points ~ 100 or higher.

MGF can be accurately calculated.