A Novel Method for Credit Portfolio Analysis - A Numerical Approximation Approach -

Mizuho-DL Financial Technology

Outline

I. What is “Analytical Method”?
II. Laplace inversion method
III. High-speed calculation of MGF

I. What is “Analytical Method”?

I. What is “Analytical Method”?

- Challenges in credit risk measurement technology
  - Fulfilling high-speed and high-accuracy at the same time
    - It is easy to satisfy either one and sacrifice the other.
  - Excluding simulation error
    - Monte Carlo (MC) simulation, which is the most familiar measurement method, inevitably involves statistical error.
  - Calculating risk contributions (RC) accurately
    - High-accuracy RC calculation method has not yet been established.
  - Handling multi-factor models
    - It is difficult to deal with multi-factor models by methods other than MC simulation.

I. What is "Analytical Method"?

- Overview of credit risk measurement approaches
  - <u>MC simulation</u>
    - Advantages: Allows for high-accuracy calculation (at the cost of spending much time)
      Applicable to multi-factor models
    - Disadvantages: Time-consuming to gain high-accuracy
      Difficult to calculate RC
  - <u>Approximate solution (IRB in Basel II, saddle-point…)</u>
    - Advantages: High-speed in general
      Can be applied to RC calculation
    - Disadvantages: Accuracy depends on portfolio
      Difficult to handle multi-factor models

I. What is "Analytical Method"?

- Choice of approach
  - There is little room for improving the two approaches.

  - In MC simulation, it is difficult to quickly calculate RC.
    - Of course, it is easy to speed up MC simulation if you can afford expensive servers…

  - In approximate solution, accuracy worsens for a certain class of portfolios (e.g. concentrated portfolios).
    - Are there any approximation formula that is highly accurate for arbitrary loss distributions?

We take the 3rd approach: numerical approximation

I. What is “Analytical Method”?

- What is numerical approximation?
  - Only numerically approximate the exact expression of loss distribution function and RC
    - “Analytical Method” developed by Mizuho-DL FT is a specific example of the numerical approximation approach
  - Free from systematic approximation based on unrealistic assumptions on portfolio (e.g. complete diversification)
  - A sophisticated calculation process enables high-speed measurement.
    - Calculation time can be substantially reduced by taking “discretization” approximation in calculation process.

➡ An approach essentially different from MC simulation and approximate solution

I. What is “Analytical Method”?

- Performance study of “Analytical Method”
  - Sample portfolios used for the study

Number of obligors	1000, 10000, 1000000
Distribution of exposures	Homogeneous Power law (Concentrated)
Number of sectors	33
Number of ratings	19

  - PC spec
    - CPU: Core^{TM}2 Duo 2.66GHz, Memory: 3.12GB
      ⇒Ordinary desktop PC

(For more details, see our paper “A novel methodology for credit portfolio analysis …”)

I. What is “Analytical Method”?

- Calculation time in “Analytical Method”
  - The following table summarizes risk measurement time in “Analytical Method”.

Table 1: Calculation time in “Analytical Method”

Number of obligors	Distribution function (including VaR and CVaR)	RC of all obligors
1,000	40sec	2min43sec
10,000	42sec	2min51sec
1,000,000	3min45sec	17min43sec

(Distribution of exposures : Concentrated)

I. What is “Analytical Method”?

- Calculation accuracy in “Analytical Method”
  - The following table summarizes risk measurement accuracy in “Analytical Method”.

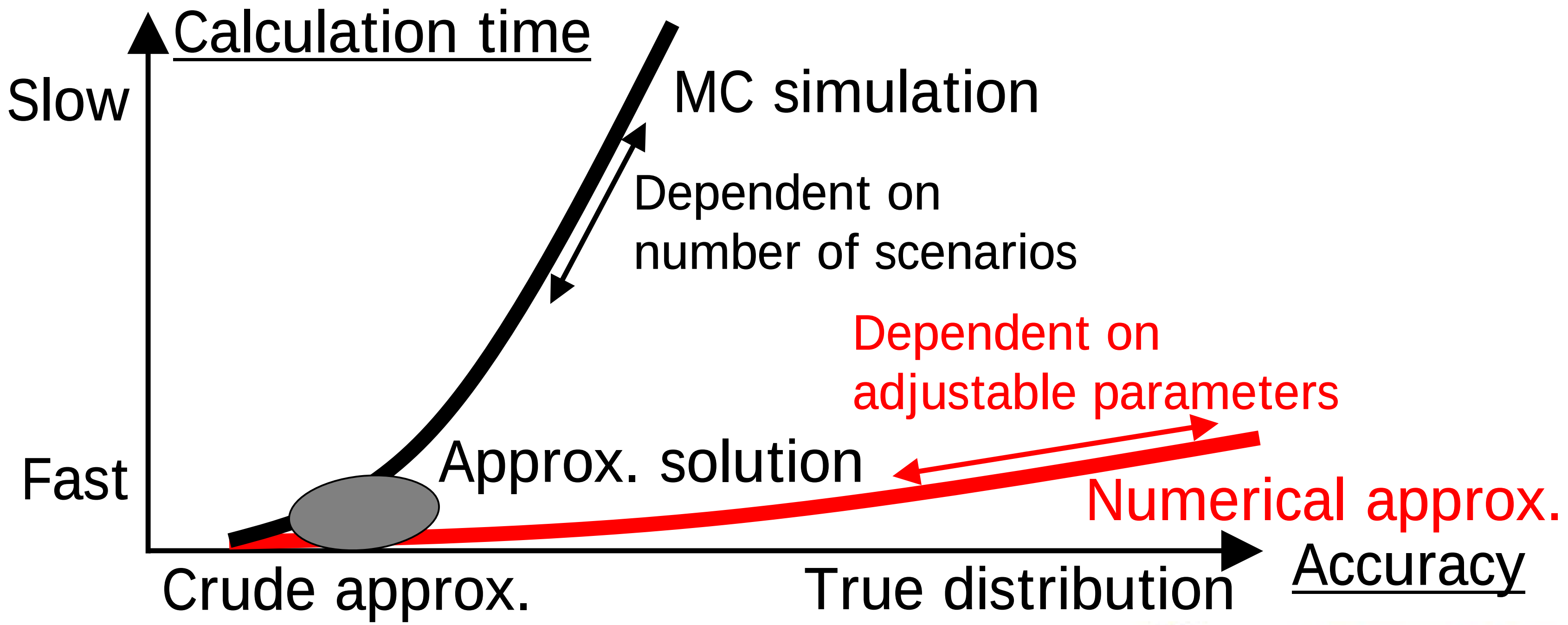
Table 2: Calculation accuracy in “Analytical Method”

Risk measure	Confidence level	Error relative to MC with 10^7 scenarios	
		Homogeneous	Concentrated
VaR	99.00%	0.32%	0.22%
	99.90%	0.31%	0.02%
	99.97%	-0.22%	-0.22%
CVaR	99.00%	0.30%	0.17%
	99.90%	-0.21%	-0.27%
	99.97%	-0.78%	-0.48%

(Number of obligors=1000000)

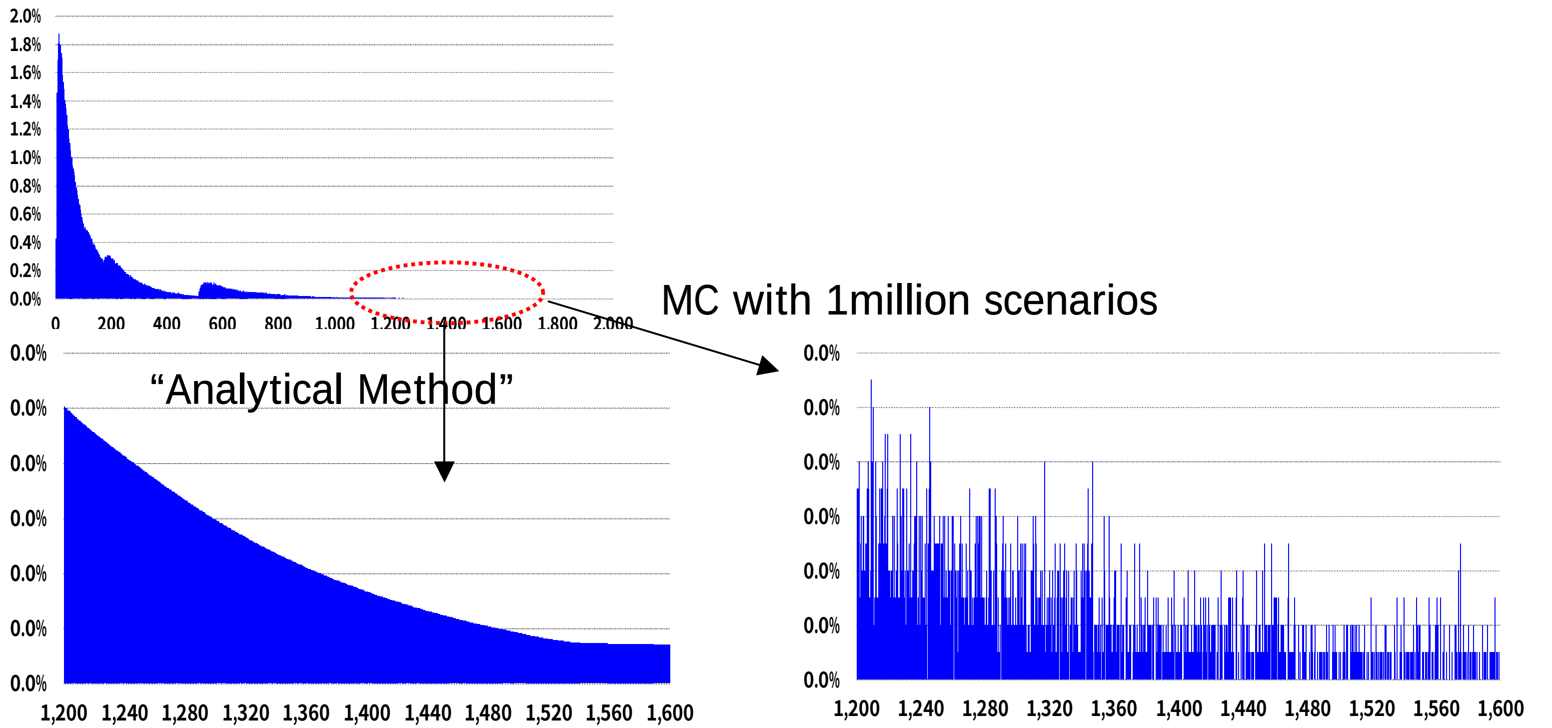
I. What is “Analytical Method”?

- Performance curve of the three approaches
  - MC simulation: Extremely steep curve
  - Approx. solution: Always fast but approximation is crude
  - Numerical approx. (∋“Analytical Method”): Gentle curve



I. What is "Analytical Method"?

- Calculation accuracy in far tail region
  - The figures below magnify the region near the 99.9% VaR of a loss distribution.



I. What is "Analytical Method"?

- Allocating risk measures into individual obligors
  - Individual VaR and CVaR contributions are defined as follows.

$$RC_i^{VaR} \equiv \frac{\partial VaR}{\partial E_i} \quad , \quad RC_i^{CVaR} \equiv \frac{\partial CVaR}{\partial E_i}$$

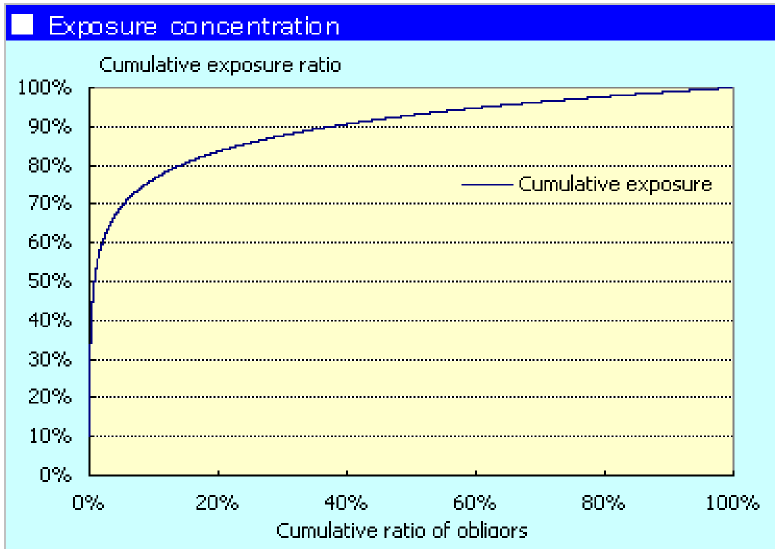
$\Rightarrow$ VaR contribution : $EC_i = E_i \cdot RC_i$

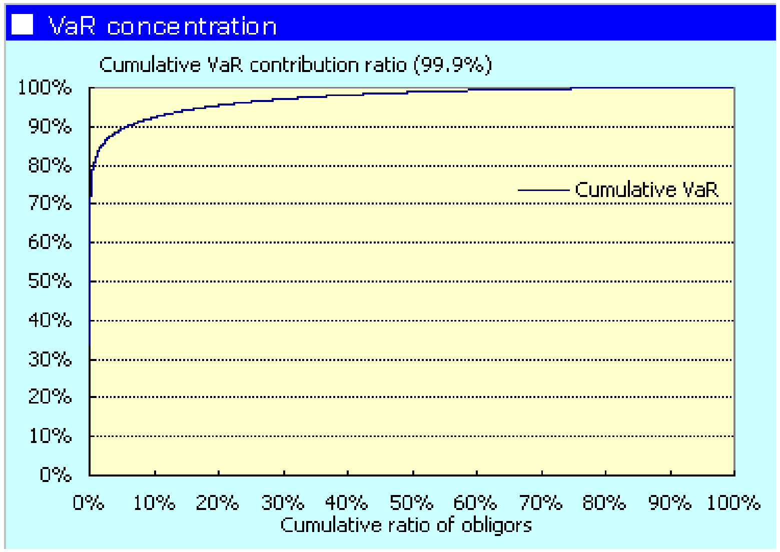
$\Rightarrow$ CVaR contribution : $EC_i^{CVaR} = E_i \cdot RC_i^{CVaR}$

  - These contributions can be accurately calculated by "Analytical Method", which provides much information about far tail region.

I. What is "Analytical Method"?

- Allocating risk measures into individual obligors
  - The graphs below show the cumulative exposures and VaR contributions (at 99.9% confidence level) of a concentrated portfolio with 10 thousand obligors.





Concentration risk can be quantitatively studied.

I. What is "Analytical Method"?

- Use of "Analytical Method" in banking practice
  - "Analytical Method" is quite effective in "comparing risk measures", because it is free from simulation error.
  - For example, impact study can be done in the following analyses without being disturbed by simulation error.
    - Stress testing for various scenarios
    - Analysis of time series variation of risk measures
    - Buying and selling simulation of loans

I. What is "Analytical Method"?

- Buying and selling simulation
  - Analyze the impact of buying a new loan on risk measures of a portfolio with 10 thousand obligors

Loan amount	: ¥10bn (0.1% of total exposure)
PD	: 1.0%

I. What is “Analytical Method”?

- Buying and selling simulation
  - The following tables show the impact of new loan.

■ MC with 1 million scenarios

Risk measure		Before buying	After buying	Difference	Rate of change
EL		170,263,758,805	170,433,816,363	170,057,558	0.100%
Standard deviation		217,798,595,633	217,752,755,326	-45,840,307	-0.021%
VaR	99.00%	931,651,036,249	931,090,171,646	-560,864,603	-0.060%
	99.90%	1,399,855,268,456	1,399,098,926,198	-756,342,258	-0.054%
	99.97%	1,746,556,960,932	1,724,735,001,649	-21,821,959,283	-1.249%
CVaR	99.00%	1,138,453,814,342	1,135,207,577,284	-3,246,237,058	-0.285%
	99.90%	1,675,291,617,710	1,658,294,760,891	-16,996,856,819	-1.015%
	99.97%	1,995,759,462,233	1,959,043,201,218	-36,716,261,015	-1.840%

■ "Analytical Method"

Risk measure		Before buying	After buying	Difference	Rate of change
EL		170,238,879,775	170,338,662,293	99,782,518	0.059%
Standard deviation		217,919,548,831	217,966,992,726	47,443,895	0.022%
VaR	99.00%	933,438,128,387	933,710,341,111	272,212,724	0.029%
	99.90%	1,399,840,868,536	1,400,281,238,247	440,369,711	0.031%
	99.97%	1,736,388,637,413	1,736,806,222,204	417,584,791	0.024%
CVaR	99.00%	1,137,086,098,100	1,137,420,522,611	334,424,511	0.029%
	99.90%	1,661,011,100,616	1,661,475,079,554	463,978,938	0.028%
	99.97%	1,962,968,916,094	1,963,448,555,622	479,639,528	0.024%

“Analytical Method” can detect a tiny difference.

I. What is “Analytical Method”?

- Formula for portfolio loss distribution
  - Probability density function (PDF) can be calculated as the Laplace inversion of moment generating function (MGF).

$$f_L(t) =$$

$$\frac{1}{2\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\alpha t}\left\{\int_{-\infty}^{\infty}\prod_{i=1}^{N}\left(1-p_i(\vec{x})+p_i(\vec{x})e^{-\alpha E_i}\right)\phi(\vec{x})d\vec{x}\right\}d\alpha$$

MGF

Laplace inversion

I. What is “Analytical Method”?

- Some crucial points in PDF calculation

  <Point 1> MGF calculation method
  - Need to quickly and accurately calculate (multi-dimensional) integral w.r.t. common risk factors

  <Point 2> Laplace inversion method
  - Need to quickly and accurately calculate integral along an infinite interval on the complex plane

“Analytical Method” is a numerical approximation for quickly and accurately executing calculations in <Point 1> and <Point 2>.

II. Laplace inversion method

II. Laplace inversion method

- Idea of Laplace inversion
  - In “Analytical Method”, Laplace inversion is done by the following two steps.

<L-1>

Represent the integral in Laplace inversion formula by an infinite series

↓

<L-2>

Calculate the infinite series using a convergence acceleration method

II. Laplace inversion method

<L-1> The Poisson method

- In the Poisson method, the infinite integral in Laplace inversion formula is approximated by an infinite series.

$$f_L(t) \cong \frac{h}{2\pi}\left[\sum_{k=-\infty}^{\infty} \exp[(\gamma + ikh)t]\hat{f}_L(\gamma + ikh)\right]$$

$$= \frac{h}{\pi}\exp(\gamma t)\cdot\left[\frac{\hat{f}_L(\gamma)}{2} + \sum_{k=1}^{\infty} \mathrm{Re}\left\{\hat{f}_L(\gamma + ikh)\exp[ikht]\right\}\right]$$

h : discretization width of integration interval

The Poisson method is formally identical with the trapezoidal formula.

II. Laplace inversion method

<L-1> The Poisson method

- The Poisson method often gives a high-accuracy approximation.
  - Error of the method when applied to distribution function

$$|err| \leq 1 - F\left(\frac{2\pi}{h} - t\right) \quad , \quad (0 < t \leq \frac{\pi}{h})$$

- This formula is valid only when an infinite number of terms are completely summed up.
- In practice, the infinite series must be truncated, while the convergence of the series is sometimes very slow.

➡ A convergence acceleration method is necessary to improve the convergence.

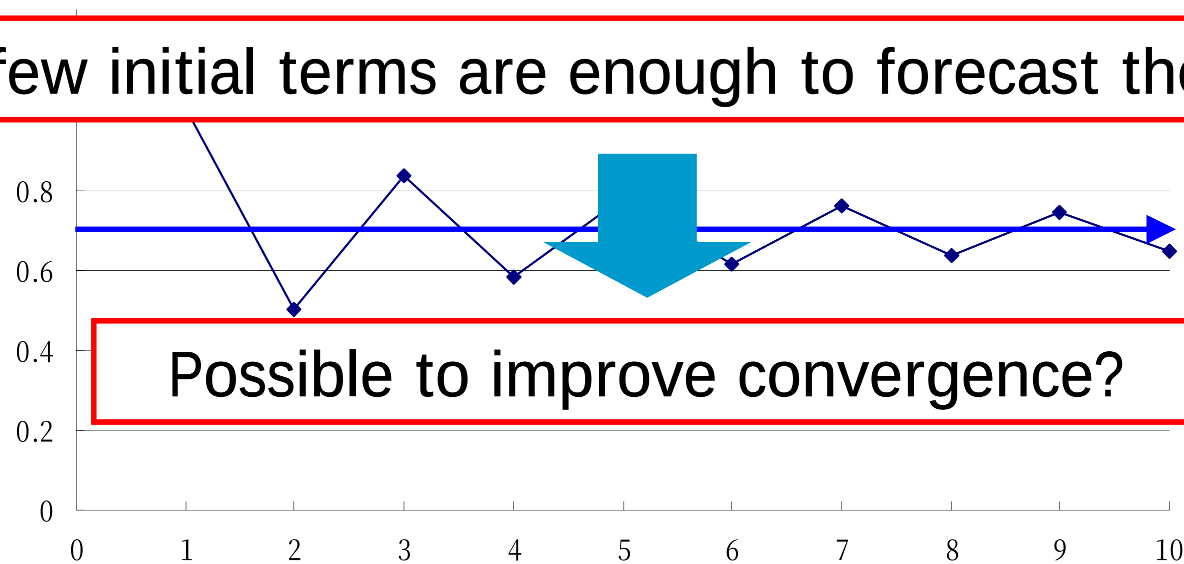
II. Laplace inversion method

- Illustration of "convergence acceleration"

  <Example: $S_n = \sum_{k=1}^{n} \frac{(-1)^k}{k}$ >

  - Sn is a very slowly converging series, but…

A few initial terms are enough to forecast the limit.



Possible to improve convergence?

II. Laplace inversion method

- <L-2> About continued fraction (CF)
  - In “Analytical Method”, the infinite series is quickly calculated using the property of continued fraction.
  - Example: Continued fraction representation of arctan(x)

Infinite series rep.: $$\arctan x = x\left\{1+\sum_{k=1}^{\infty}\frac{(-1)^k x^{2k}}{2k+1}\right\}$$

Continued fraction rep.: $$\arctan x = \cfrac{x}{1+\cfrac{x^2}{3+\cfrac{(2x)^2}{5+\cfrac{(3x)^2}{7+\cdots}}}}$$

II. Laplace inversion method

- <L-2> The property of continued fraction
  - Continued fraction representation sometimes converges extremely faster than infinite series representation does.

<Example: Numerical calculation of **π**>

Calculate an approximate value of **π** from **π**=4arctan(1)

π=<u>3.1415</u><u>9</u>36… (infinite series with <u>1 million terms</u>)

π=<u>3.1415926535898</u>… (<u>18th order</u> continued fraction)

II. Laplace inversion method

- <L-2> Convergence acceleration with CF
  - Convergence is expected to drastically improve if the infinite series can be converted to a continued fraction.

$$f_L(t) \cong \frac{h}{\pi}\exp(\gamma t)\sum_{k=0}^{\infty}\mathrm{Re}\left(c_k z^k\right)$$

$$= \frac{h}{\pi}\exp(\gamma t)\times[a_0 + \cfrac{a_1 z}{1+\cfrac{a_2 z}{1+\cfrac{a_3 z}{1+\cfrac{a_4 z}{1+\cdots}}}}]$$

Need to calculate $\{a_0, a_1, \cdots\}$, given the values of $\{c_0, c_1, \cdots\}$

II. Laplace inversion method

- <L-2> QD algorithm
  - QD algorithm is a method for efficiently calculating the coefficients $\{a_0, a_1, \cdots\}$ of a continued fraction from the coefficients $\{c_0, c_1, \cdots\}$ of the corresponding infinite series.

Input: $e_0^{(i)} = 0 \quad , \quad q_1^{(i)} = \dfrac{c_{i+1}}{c_i}$

Recursion: $e_k^{(i)} = e_{k-1}^{(i+1)} + q_k^{(i+1)} - q_k^{(i)} \quad , \quad q_{k+1}^{(i)} = \dfrac{q_k^{(i+1)} \cdot e_k^{(i+1)}}{e_k^{(i)}}$

Output: $a_{2k} = -e_k^{(0)} \quad , \quad a_{2k-1} = -q_k^{(0)}$

II. Laplace inversion method

- <L-2> Convergence acceleration with CF
  - Loss PDF can be quickly calculated by applying the property of continued fraction to Laplace inversion.

<The result of the Poisson method>

$$f(t) \cong \frac{h}{\pi}\exp(\gamma t)\cdot\left[\frac{\hat{f}(\gamma)}{2}+\sum_{k=1}^{\infty}\mathrm{Re}\left\{\hat{f}(\gamma+ikh)\exp[ikht]\right\}\right]$$

Can be converted to a continued fraction by using QD algorithm with $c_k = \hat{f}_L(\gamma+ikh)$, $z=\exp(iht)$

II. Laplace inversion method

- <L-2> Calculation of loss distribution
  - Once the coefficients $\{a_0, a_1, \cdots\}$ are obtained, the values of density function at arbitrary losses can be calculated.

$$f(t) \cong \frac{h}{\pi} \exp(\gamma t) \cdot [a_0 + \cfrac{a_1 z}{1 + \cfrac{a_2 z}{1 + \cdots}}]$$

$$z = \exp(iht)$$

Since $\{a_0, a_1, \cdots\}$ does not depend on loss t, the whole distribution can be quickly calculated.

II. Laplace inversion method

- Summary of Laplace inversion method
  - In “Analytical Method”, Laplace inversion is quickly and accurately done by the following two steps.

<L-1> Create an infinite series by the Poisson method

⇒The Poisson method is formally identical with the trapezoidal formula.

<L-2> Convergence acceleration by QD method

⇒QD method is an algorithm for continued fraction transformation.

III. High-speed calculation of MGF

III. High-speed calculation of MGF

- Conditions necessary for high-speed calculation

  <u>Conditionally independent</u>

  - Conditional on fixed risk factors, default occurs independently.

  <u>1-factor decomposable</u>

  - Portfolio consists of a small number of sub-portfolios, each of which is described by a 1-factor model.

These properties are the keys to high-speed calculation.

III. High-speed calculation of MGF

- Conditional independence
  - "Analytical Method" can be applied to credit risk measurement based on conditional independence model.
  - For example, the multi-factor firm-value model below satisfies conditional independence.

$$Z_i = \sum_{j=1}^{N_F} \alpha_{ij} X_j + \sqrt{1 - \sum_{j=1}^{N_F} \alpha_{ij}^2}\, \varepsilon_i$$

Firm-value

Common risk factor

Idiosyncratic factor

Sensitivity coefficient

III. High-speed calculation of MGF

- 1-factor decomposition
  - In a standard practice, sensitivity coefficients are estimated only for “sectors”, not for individual obligors.

$$\alpha_{ij} \quad \Rightarrow \quad \alpha_{S(i)\,j} \qquad S(i)$$: sector of obligor i

  - In this case, the firm-value takes the form

$$Z_i = \beta_{S(i)} \tilde{Y}^{S(i)} + \sqrt{1 - \beta_{S(i)}{}^2}\,\tilde{\varepsilon}_i$$

$$\beta_{S(i)} = \sqrt{\sum_{j=1}^{N} \alpha_{S(i)\,j}{}^2} \quad , \quad \tilde{Y}^{S(i)} = \frac{1}{\beta_{S(i)}} \sum_{j=1}^{N} \alpha_{S(i)\,j} \tilde{X}_j$$: sectoral factor

⇒1-factor decomposable

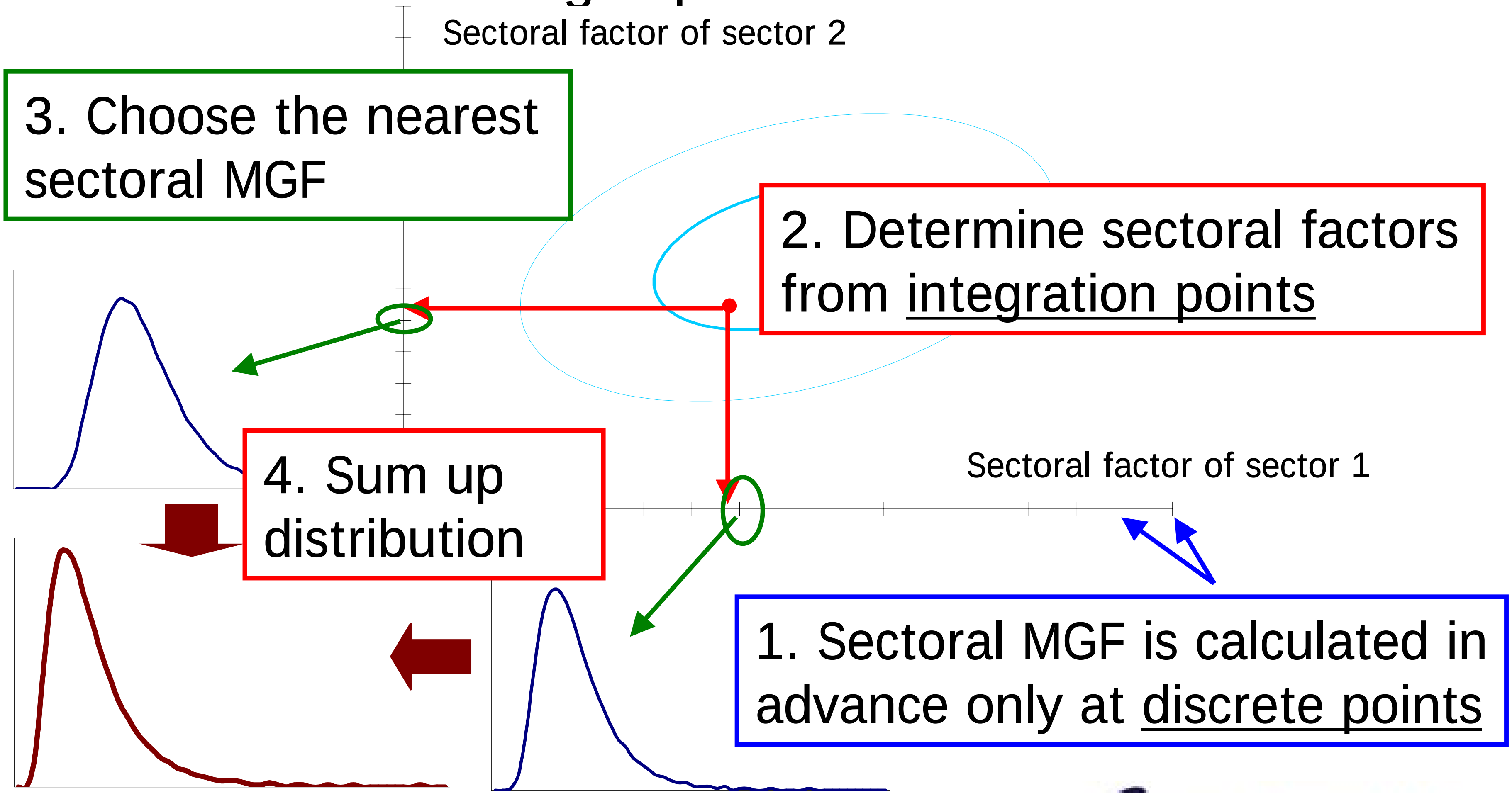
III. High-speed calculation of MGF

- Idea of high-speed calculation of MGF
  - It is time-consuming to calculate MGF because it requires a loop proportional to the number of obligors.
    ⇒This calculation is done for each sector on 1 dimension.
  - Correlation structure is described among sectors.
    ⇒High-dimensional calculation is necessary to take into account correlation.

Split marginal distribution calculation and correlation structure calculation

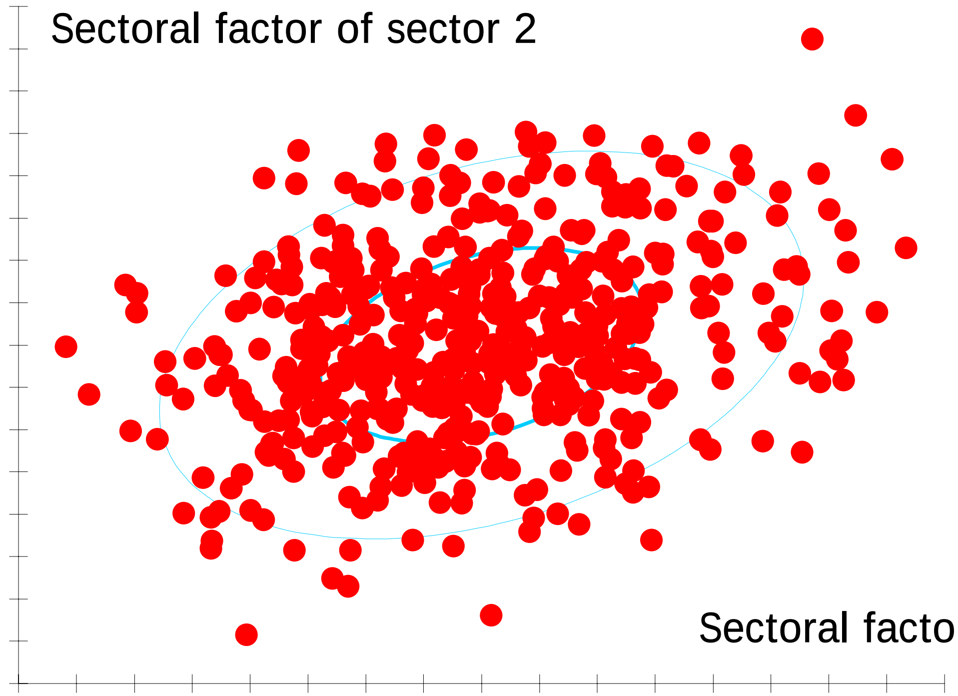
III. High-speed calculation of MGF

- Illustration of high-speed calculation of MGF



III. High-speed calculation of MGF

- Illustration of high-speed calculation of MGF



5. Repeat 1.~4. for all integration points

III. High-speed calculation of MGF

- Computational complexity of high-speed calculation
  - Computational complexity of a naive method for calculating MGF is given by

    (Number of obligors) X (Number of integration points).
    $\sim O(10^{4\text{-}6})$ $\sim O(10^{6})$

  - Computational complexity of the high-speed calculation is given by
    $\sim O(10^{4\text{-}6})$ $\sim O(10^{2})$
    (Number of obligors) X (Number of discrete points)
    + (Number of integration points) X (Number of sectors).
    $\sim O(10^{6})$ $\sim O(10^{0\text{-}2})$

Computational complexity can be drastically reduced by the high-speed calculation.

III. High-speed calculation of MGF

- Accuracy of high-speed calculation
  - In the high-speed calculation, the continuous Euclidean space spanned by sectoral factors is replaced in effect with a hypercubic discrete lattice.
  - The error due to this approximation can be controlled, and becomes small as the number of discrete points (which specifies the fineness of the lattice) increases.
  - We have confirmed that this approximation was sufficiently accurate if the number of discrete points ~ 100 or higher.

$\Rightarrow$ $\boxed{\text{MGF can be accurately calculated.}}$